Descriptive Set Theory
Lecture 17

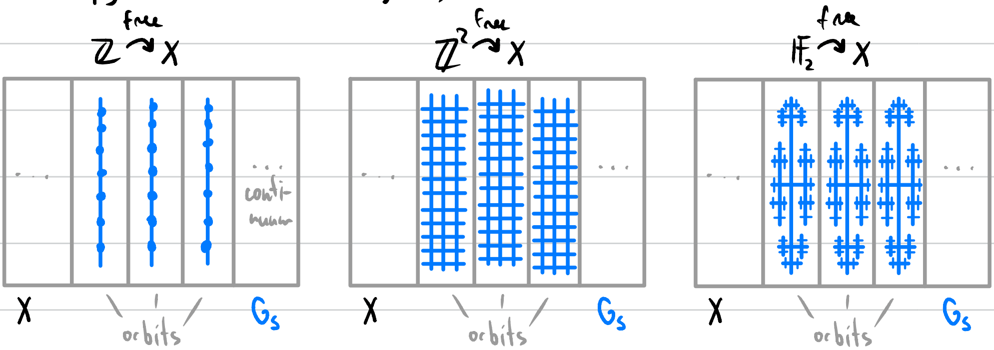
Ergodicity and ergodic theory (:= the study of transformation or (semi)groups of them on measure spaces, typically probability) have applications to a 27-year-old subject called **descriptive combinatorics**, which deals with Borel (or more generally, **definable**) graphs on Polish spaces and tries to understand colouring / matching question about them with regularity conditions on these objects (e.g. a Borel colouring).

A **Borel graph** on a Polish space X is a Borel subset $G \subseteq X^2$ s.t. it's symmetric and irreflexive. A graph is **locally ctbl/finite** if each vertex has ctbly/finitely many neighbours. Locally ctbl Borel graphs arise from Borel actions $\Gamma \curvearrowright X$ of ctbl groups Γ.

Let $\Gamma \curvearrowright X$ be a Borel action and let S be a symmetric generating set for Γ. We define the **Schreier graph G_S** of this action wrt S as follows: $\forall x, y \in X$,

$$(x,y) \in G_S :\Longleftrightarrow y = \gamma \cdot x \text{ for some } \gamma \in S.$$

The connected components of G_S are exactly the orbits of the action $\Gamma \curvearrowright X$. If the action is free (i.e. no nonidentity group element has a fixed point) then each component of G_S is a copy of the Cayley graph of Γ wrt S.



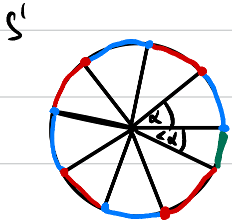
For example, the Hamming graph on $2^{\mathbb{N}}$ is exactly the Schreier graph of the group $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ with the generating set $\{\gamma_n : n\in\mathbb{N}\}$, where $\gamma_n := (0,0,\dots,0,\underset{n}{1})$ and γ_n acts on $2^{\mathbb{N}}$ via flipping the n^{th} bit, so the action is continuous.

Now let G be the Schreier graph of an irrational rotation $\mathbb{Z} \curvearrowright S^1$ by the rotation T_α. Since each component is a $\mathbb{Z}$-line, we can choose a starting point in each component via Axiom of Choice and obtain a 2-colouring of G, so the chromatic number of G is 2, $\chi(G) = 2$.

What is the Borel / measurable / Baire measurable chromatic number of G? (χ_B, χ_λ, χ_{BM})

Clearly, $2 = \chi(G) \le \begin{matrix} \chi_\lambda(G) \\ \chi_{BM}(G) \end{matrix} \le \chi_B(G)$.

Thus, $\chi_B(G) \le 3$.

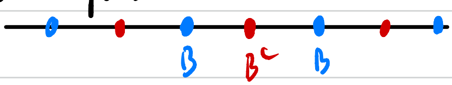


The following is a consequence of (generic) ergodicity:

Prop. $\chi_\lambda(G) = \chi_{BM}(G) = 3 = \chi_B(G)$.

Proof. Suppose there is a measurable 2-colouring, i.e. colours are measurable sets B and B^c. Note that B is an invariant set for the rotation $T_{2\alpha}$ by 2α, which is still irrational, so the action is still ergodic. Thus B is null or conull. But $\lambda(B) = \lambda(B^c) = \frac{1}{2}$ because $T_\alpha(B) = B^c$ and rotation preserves measure, contradiction.

Each component:



Same works for a BM colour B because $T_\alpha(B) = B^c$ and T_α is a homeomorphism so B is meager $\Leftrightarrow B^c$ is meager. □

Borel sets and hierarchy.

A **measurable space** is a pair $(X, \mathcal{S})$ where X is a set and $\mathcal{S}$ is a σ-algebra on it.

Examples. For a Polish space X, $(X, \mathcal{B}(X))$, $(X, BM(X))$, & $(X, MEAS_\mu(X))$ for some Borel measure μ on X.

For a given σ-alg. $\mathcal{S}$ on X, we say that a $\mathcal{E} \subseteq \mathcal{P}(X)$ **generates** $\mathcal{S}$ if $\mathcal{E} \subseteq \mathcal{S}$ & $\mathcal{S}$ is the smallest σ-alg. containing $\mathcal{E}$. For $\mathcal{E} \subseteq \mathcal{P}(X)$, let $\neg \mathcal{E} := \{A^c : A \in \mathcal{E}\}$.

Prop. If $\mathcal{E}$ generates a σ-alg. $\mathcal{S}$ and $\mathcal{S}' \supseteq \mathcal{E}, \neg\mathcal{E}$ & $\mathcal{S}' \subseteq \mathcal{S}$ is closed under ctbl unions & ctbl intersections, then $\mathcal{S}' = \mathcal{S}$.

Proof. Let $\mathcal{S}'' := \{A \in \mathcal{S}' : A, A^c \in \mathcal{S}'\}$, so $\mathcal{S}'' \subseteq \mathcal{S}' \subseteq \mathcal{S}$, & we show that $\mathcal{S}'' = \mathcal{S}$. By def, $\mathcal{E} \subseteq \mathcal{S}''$ & $\mathcal{S}''$ is closed under complements. Let $(A_n) \subseteq \mathcal{S}''$. Then $\bigcup_n A_n \in \mathcal{S}'$ & $(\bigcup_n A_n)^c = \bigcap_n A_n^c \in \mathcal{S}'$ because $A_n^c \in \mathcal{S}'$. Thus, $\mathcal{S}''$ is a σ-alg. containing $\mathcal{E}$. $\square$

For measurable spaces $(X, \mathcal{A})$ & $(Y, \mathcal{B})$, a function $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ is called **measurable** if $f^{-1}(\mathcal{B}) \subseteq \mathcal{A}$. If Y is a top space, then by default we turn it into a measurable space $(Y, \mathcal{B}(Y))$. Thus, a function $f : X \to Y$ is called **$\mathcal{A}$-measurable** if it is measurable a function from $(X, \mathcal{A})$ to $(Y, \mathcal{B}(Y))$. This equivalent to preimages of open sets being in $\mathcal{A}$ because of the following:

Prop. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces and $\mathcal{E}$ is a generating set for $\mathcal{B}$. Then for any $f: X \to Y$, if $f^{-1}(\mathcal{E}) \subseteq \mathcal{A}$ then f is measurable.

Proof. Let $\mathcal{B}' := \{B \in \mathcal{B} : f^{-1}(B) \in \mathcal{A}\}$. Then $\mathcal{B}' \supseteq \mathcal{E}$ and $\mathcal{B}'$ is a σ-algebra because $\mathcal{A}$ is, so $\mathcal{B}' = \mathcal{B}$. $\square$

Def. Let X be a top. space. We define the Borel hierarchy as follows: $\Sigma^0_1(X) :=$ the collection of open sets. For any ctbl ordinal α, define $\Pi^0_\alpha(X) := \neg \Sigma^0_\alpha(X)$. Suppose $\Pi^0_\beta(X)$ is defined for all $\beta < \alpha$, $\alpha, \beta < \omega_1$, and define

$$\Sigma^0_\alpha(X) := \{\bigcup_n A_n : A_n \in \Pi_{\beta_n} \text{ for } \beta_n < \alpha\}.$$

Also, $\Delta^0_\alpha(X) := \Sigma^0_\alpha(X) \cap \Pi^0_\alpha(X)$.

Prop. For any metrizable X, the picture is:

$$\underbrace{\begin{matrix} & \Sigma^0_1 & \subseteq & \Sigma^0_2 & & & \Sigma^0_\alpha & & \\ \Delta^0_1 & & \subseteq & \Delta^0_2 & \subseteq \cdots & \Delta_\alpha & & \Delta^0_{\alpha+1} & \cdots \\ & \Pi^0_1 & \subseteq & \Pi^0_2 & & & \Pi^0_\alpha & & \end{matrix}}_{\mathcal{B}(X)} \qquad \omega_1$$

(a) $\forall\, \beta < \alpha < \omega_1$, $\quad \Delta^0_\beta(X) \subseteq \begin{matrix} \Sigma^0_\beta(X) \\ \Pi^0_\beta(X) \end{matrix} \subseteq \Delta^0_\alpha(X)$.

(b) $\mathcal{B}(X) = \bigcup\limits_{\alpha<\omega_1} \Delta^0_\alpha(X) = \bigcup\limits_{\alpha<\omega_1} \Sigma^0_\alpha(X) = \bigcup\limits_{\alpha<\omega_1} \Pi^0_\alpha(X)$.

Proof. (a) Enough to prove $\Sigma^0_\beta \subseteq \Delta^0_\alpha$ because Δ^0_α is closed under complements. For $\Sigma^0_\beta \subseteq \Delta^0_\alpha$, the part $\Sigma^0_\beta \subseteq \Pi^0_\alpha$ is by def, so we show $\Sigma^0_\beta \subseteq \Sigma^0_\alpha$.
For $\beta = 1$ & $\alpha = 2$, this is just the statement that open sets are F_σ, which true in metrizable spaces.
For $\alpha > 2$, $\beta = 1$, we have $\Sigma^0_1 \subseteq \Pi^0_2 \subseteq \Sigma^0_\alpha$.
For $\beta \geq 2$, then Σ^0_β itself is a union of previous Π-sets, and so is Σ^0_α, thus $\Sigma^0_\beta \subseteq \Sigma^0_\alpha$.

(b) All equalities except the first follow from (a). For the first one, $\supseteq$ is obvious (technically, by induction on ω_1) & $\subseteq$ follows from the regularity of ω_1, which implies that $\bigcup\limits_{\alpha<\omega_1} \Sigma^0_\alpha(X)$ is closed under ctbl unions:
if $\alpha_n < \omega_1$ & $A_n \in \Sigma^0_{\alpha_n}$ then $\alpha := \sup\limits_n \alpha_n$ is still $< \omega_1$ so all $A_n \in \Sigma^0_\alpha$, hence $\bigcup\limits_n A_n \in \Sigma^0_{\alpha+1}$.
So $\bigcup\limits_{\alpha<\omega_1} \Sigma^0_\alpha = \bigcup\limits_{\alpha<\omega_1} \Delta^0_\alpha$ is a σ-algebra containing Σ^0_1. $\square$